PHYS 705: Classical Mechanics

HOMEWORK COMMENTS

Indices and Derivatives again:

HW #2: 1.10

$q_i = q_i\left(s_j, t\right)$ Use different indices for q_i and s_j

especially, when evaluating $$\frac{\partial L}{\partial s_i} = \frac{\partial L}{\partial q_j}\frac{\partial q_j}{\partial s_i} + \frac{\partial L}{\partial \dot{q}_j}\frac{\partial \dot{q}_j}{\partial s_i} + \frac{\partial L}{\cancel{\partial t}}\frac{\cancel{\partial t}}{\partial s_i}$$

$\dot{q}_i = \dot{q}_i\left(s_j, \dot{s}_j, t\right)$ not just $\dot{q}_i = \dot{q}_i\left(\dot{s}_j, t\right)$

check $$\frac{d}{dt}\left[q_i\left(s_j,t\right)\right] = \frac{\partial q_i}{\partial s_k}\left(s_j,t\right)\dot{s}_k + \frac{\partial q_i}{\partial t}\left(s_j,t\right)$$

Also,

$$\frac{\partial q_i\left(s_j,t\right)}{\partial \dot{s}_k} = 0 \quad but \quad \frac{\partial \dot{q}_i\left(s_j,\dot{s}_j,t\right)}{\partial \dot{s}_k} \neq 0 \quad and \quad \frac{\partial \dot{q}_i\left(s_j,\dot{s}_j.t\right)}{\partial s_k} \neq 0$$

HW #2: 1.19

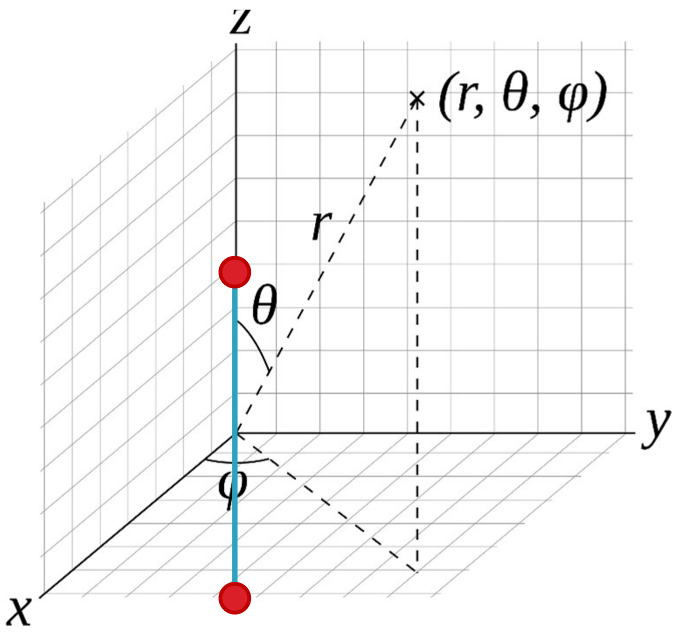
Choice of coordinates and convention for *U* (gravitational potential eng)

→ *U* should decrease as the mass decreases in height

$U = mgl \rightarrow$

$U = 0 \rightarrow$

$U = -mgl \rightarrow$



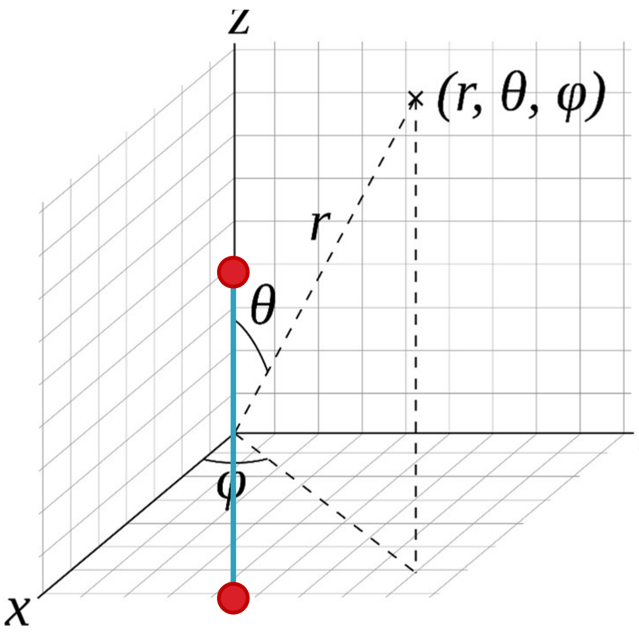
$z = l\cos\theta$

$U = mgl\cos\theta$

HW #2: 1.19

Choice of coordinates and convention for U (gravitational potential eng)

→ One can also choose



$U = 2mgl \rightarrow$

$U = mgl \rightarrow$

$U = 0 \rightarrow$

$z = l\cos\theta$

$U = mgl\left(1 + \cos\theta\right)$

Reference point for *U* is different

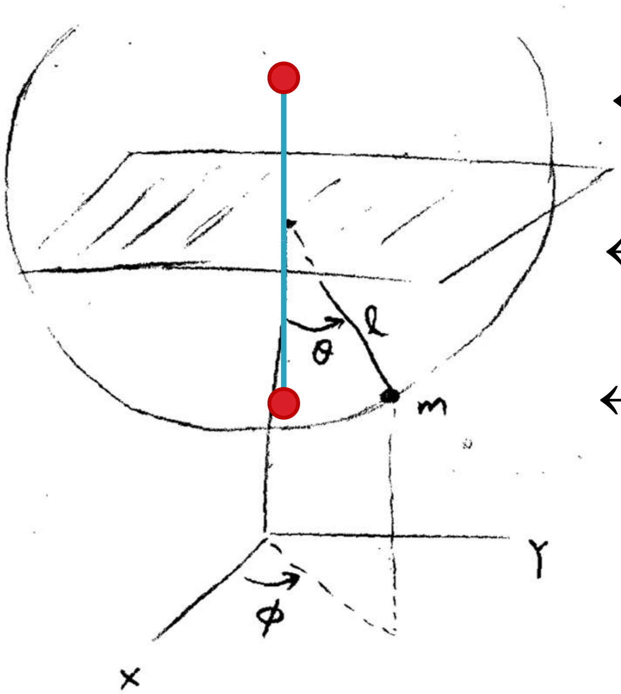
HW #2: 1.19

Choice of coordinates and convention for U (gravitational potential eng)

→But this pair of z and U definitions are not physically consistent

$$z = -l\cos\theta$$

$$U = mgl\left(1+\cos\theta\right)$$



$U = 0 \rightarrow$

$U = mgl \rightarrow$

$U = 2mgl \rightarrow$

Not Ok

$\leftarrow z = l$

$\leftarrow z = 0$

$\leftarrow z = -l$

Ok

Not standard convention: U increases as m decreases in height

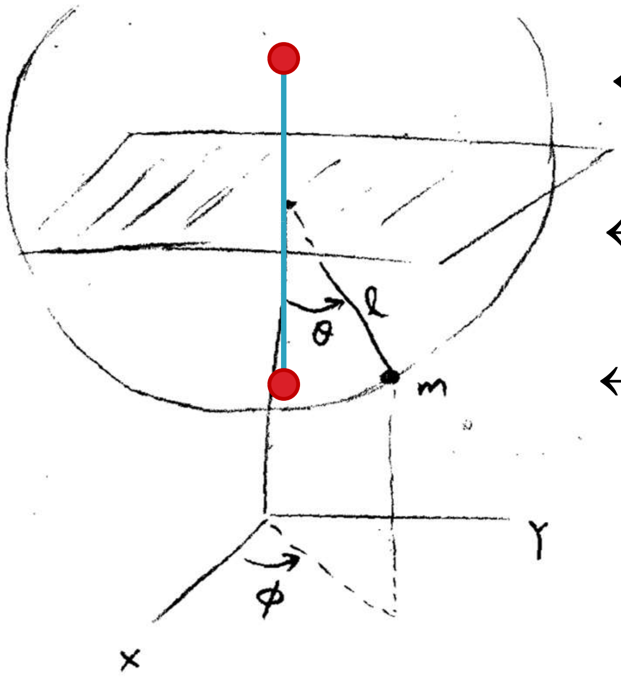
HW #2: 1.19

Choice of coordinates and convention for U (gravitational potential eng)

→This pair of *z* and *U* is ok.

$z = -l\cos\theta$

$U = -mgl\cos\theta$



$U = +mgl \rightarrow$ $\leftarrow z = l$

$U = 0 \rightarrow$ $\leftarrow z = 0$

$U = -mgl \rightarrow$ $\leftarrow z = -l$

Ok

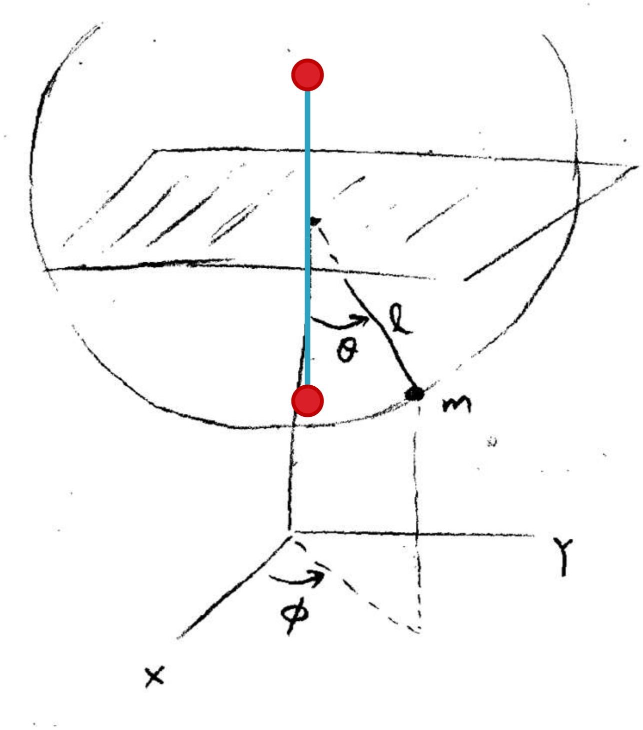
Ok

HW #2: 1.19

Always draw a picture to indicate how your generalized coordinates related to your system

$U = 0$

$z = 0 \rightarrow$



HW #2: 1.19

Don’t forget the product rule:

$ml^2 \sin^2\theta\,\dot{\phi}$ is a function of two variables: θ and ϕ

So,

$$\frac{d}{dt}\left(ml^2 \sin^2\theta\,\dot{\phi}\right) = ml^2 \sin^2\theta\,\ddot{\phi} + 2ml^2 \sin\theta \cos\theta\dot{\theta}\dot{\phi}$$

LECTURE REVIEW

Variational Calculus: N indep (proper) generalized coordinates (with constraint explicitly included)

Everything proceeds as before, and we get:

$$\frac{dI}{d\alpha} = \int_{x_A}^{x_B} \sum_i \left[\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i{}'} \right) \right] \eta_i(x)\, dx = 0$$

If all the variations $\eta_i(x)$ are independent, this equation requires that each coefficients in the integrant must vanish independently. Then, we have,

$$\frac{\partial F}{\partial y_i} - \frac{d}{dx}\left(\frac{\partial F}{\partial y_i{}'} \right) = 0 \quad i = 1, 2, \cdots, N$$

This is the Euler-Lagrange Eq for a proper set of generalized coords.

Variational Calculus: N improper generalized coordinates with M constraints

In general with *N dependent* variables and *M* constraints, we can write the E-L equation with *M* Lagrange undetermined multipliers as,

$$\begin{cases} \dfrac{\partial F}{\partial y_i} - \dfrac{d}{dx}\left(\dfrac{\partial F}{\partial y_i\,'}\right) - \displaystyle\sum_{k=1}^{M} \lambda_k(x)\frac{\partial g_k}{\partial y_i} = 0 & i = 1, 2, \cdots, N \quad \text{(for stationarity)} \\ g_k(y_i; x) = 0 & k = 1, 2, \cdots, M \quad \text{(to be on constraints)} \end{cases}$$

Here, we have *N*+*M* unknowns: $y_i(x)$ and $\lambda_k(x)$ “Lagrange undetermined multiplier”

And, we have *N*+*M* equations: top (*N*) and bottom (*M*)

The magnitude of the constraint force for y_i can also be calculated as:

$$Q_i = \sum_{k=1}^{M} \lambda_k(x)\frac{\partial g_k}{\partial y_i}$$

Hamilton's Principle

In Classical Mechanic, the dynamical quantity that we want to extremize is the **Action**,

$$I = \int_{1}^{2} L\left(q_i, \dot{q}_i, t\right) dt$$

where $L = T - U$ is the Lagrangian of the system

(Note: Here we don't require all the generalized coordinates q_i to be necessarily independent. The $q_i\,'s$ can be linked through constraints.)

Hamilton's Principle: The motion of a system from t_1 to t_2 is such that the action evaluated along the actual path is *stationary*.

Lagrange Equation of Motion

So, we apply our variational calculus results to the action integral.

We also further assume that the system is **monogenic**, i.e., all forces except forces of constraint are derivable from a potential function which can be a function of $q_i, \dot{q}_i$, and t

The resultant equation is the Lagrange Equation of Motion with N generalized coordinates (not necessary proper) and M Holonomic constraints:

$$\begin{cases} \dfrac{\partial L}{\partial q_i} - \dfrac{d}{dt}\left(\dfrac{\partial L}{\partial \dot{q}_i}\right) - \displaystyle\sum_{k=1}^{M} \lambda_k(t) \dfrac{\partial g_k}{\partial q_i} = 0 \\ g_k(q_i; t) = 0 \end{cases} \qquad \begin{array}{l} i = 1, 2, \cdots, N \\ k = 1, 2, \cdots, M \end{array}$$

Forces of Constraint

Comments:

1. The Lagrange EOM can be formally written as:

$$\frac{\partial L}{\partial q_i}-\frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_i}\right)=\sum_{k=1}^{M}\lambda_k(t)\frac{\partial g_k}{\partial q_i}=Q_i \qquad i=1,2,\cdots,N$$

where the Q_i are the generalized forces which give the magnitudes of the forces needed to produce the individual constraints.

- the generalized coordinates q_i are NOT necessarily independent and they are linked through the constraint equations.

- since the choice of the sign for λ_k is arbitrary, the direction of the forces of constraint forces cannot be determined.

Proper Generalized Coordinates

Comments:

2. If one chooses a set of "proper" (*independent*) generalized coordinates in which the (*N*-*M*) q_j's are no longer linked through the constraints, then the Lagrange EOM reduces to:

$$\frac{\partial L}{\partial q_j} - \frac{d}{dt}\left(\frac{\partial L}{\partial \dot{q}_j}\right) = 0 \qquad j = 1, 2, \cdots, N - M$$

- In practice, one typically will explicitly use the constraint equations to reduce the number of variables to the (*N*-*M*) proper set of generalized coordinates.
- However, one CAN'T solve for the forces of constraint here